Differential equation exercise

The change in the cost C of ordering and holding, as the quantity x changes, is given by $\dfrac{dC}{dx} = a - \frac{C}{x}$, where a is a constant. Find $C = f(x)$ if $C = C_0$ when $x = x_0$.

Solution

It is a homogeneous differential equation (with two homogeneous functions of degree 0):

$$dC = (a - \frac{C}{x})dx$$

$$dC + (\frac{C}{x} - a)dx = 0$$

The degree of homogeneity is 0. I propose the substitution $u = C/x$ and then $c = ux$, $dc = xdu + udx$.

$$xdu + udx + (u - a)dx = 0$$

$$xdu + udx + udx - adx = 0$$

$$(2u - a)dx = -xdu$$

$$\frac{dx}{x} = \frac{du}{a - 2u}$$

We integrate both sides:

$$ln(x) = \frac{ln(a - 2u)}{-2} + C$$

Substituting back:

$$ln(x) = \frac{ln(a - 2(C/x))}{-2} + C$$

Solving for C:

$$-2ln(x) - K = ln(a - 2C/x)$$

$$e^{-2ln(x)-K} = a - 2C/x$$

$$\frac{-x}{2}x^{-2}H + \frac{ax}{2} = C$$

$$-\frac{1}{2x}H + \frac{ax}{2} = C$$

We see the particular solution:

$$-\frac{1}{2x_0}H + \frac{ax_0}{2} = C_0$$

Solving for H:

$$H = (\frac{ax_0}{2} - C_0)(2x_0)$$

$$H = ax_0^2 - 2x_0C_0$$

Substituting back:

$$-\frac{1}{2x}\left[ax_0^2 - 2x_0C_0\right] + \frac{ax}{2} = C$$

$$C = \frac{ax}{2} + \frac{1}{x}\left(C_0x_0 - \frac{ax_0^2}{2}\right)$$